THE TORTUOUS WAYS TOWARD A NEW UNDERSTANDING OF ALGEBRA IN THE ITALIAN ABBACUS SCHOOL (14^{TH}–16^{TH} CENTURIES)

Jens Høyrup

Roskilde University

Algebra as we encounter it in Stifel (1544) or Descartes (1637) looks wholly different from what we know from al-Khwārizmī and Fibonacci. Indeed, early Modern algebra did not build on these: its foundation was the algebra of the Italian Abbacus school. The paper follows the development of this tradition from 1307 onward, in particular the appearance of abbreviations, the naming of powers and roots, formal calculations, schemes, and the solution of higher-degree equations.

THE TRANSFORMATION

Ancient Babylonian and Ancient Egyptian mathematics were powerful calculational tools for the solution of scribal tasks - accounting, planning of resources, measurement of land; they were developed and taught for that purpose. What else was achieved by them – e.g., the impressive feats of Old Babylonian "Algebra" – was derivative and secondary to that purpose.

Classical Ancient mathematics had many components:[1]

- "Practical mathematics" of the scribal kind.
- "Liberal-Arts"-mathematics, the kind of mathematics which a well-bred person ought to know about – which was generally very little.
- What is mostly thought of as "Greek mathematics", the theoretical geometry of Euclid, Archimedes, Apollonios etc.

The latter type (though only a minor segment of it) turned out to be a powerful tool in Ptolemaic astronomy and in theoretical static and optics; Hero was also able to apply a small part to mensuration of the "scribal" type.[2] However, this was not the main purpose for which it was created, and until the late Renaissance it did not significantly broaden the range of applications it could serve.

Modern mathematics as it has unfolded since around 1600 has turned out to be an immensely more powerful tool for an ever-increasing range of practical objectives. What enabled it to go beyond the limits of ancient theoretical mathematics was the introduction of symbolic, formal calculation - first in algebra, then in *analysis infinitorum*, then in the calculus of probabilities and theoretical statistics, etc.

[1] See [Cuomo 2001]. The occasional lack of precision of this book does not prevent it from being an excellent introduction to the diversity of Classical mathematics.

[2] A little bit, though even less, also crept into for instance *Geometrica* and *Stereometrica*, pseudo-Heronian compilations closer to scribal traditions.

Algebra was thus the decisive stimulus – yet not the kind of algebra which Europe had known from al-Khwārizmī and Fibonacci. This kind, indeed, could never have transformed mathematics as a whole. What was the difference? And what had happened to algebra?

NESSELMANN'S CATEGORIES, AN ONLY PARTIAL ANSWER

A first approach to the difference would make use of the three-stage scheme which Nesselmann proposed in his *Algebra der Griechen* [1842: 302]. A "first and lowest" stage in the development of algebra should be that of "rhetorical algebra", which expresses everything in full words.[3] Nesselmann's second stage is "syncopated algebra"; here, standard abbreviations are used for certain recurrent concepts and operations, even though "its exposition remains essentially rhetorical" – that is, the whole exposition can be expanded into full words. The third stage is "symbolic algebra"; here, "all forms and operations that appear are represented in a fully developed language of signs that is completely independent of the oral exposition".

It is obvious that al-Khwārizmī's and Fibonacci's algebras are rhetorical, and no less obvious that Descartes' algebra is symbolic. However, Nesselmann's notion of symbolic algebra is broader than we might at first expect. He does indeed take European mid-17th-century algebra to be symbolic, but also counts the Indian use of schemes to the same category. He shows no examples of this, but we may borrow one from Bhāskara II as transcribed in [Datta & Singh 1962: II, 32]

yâ gha 8	*yâ va* 4	*kâ vayâ.bhâ* 10
yâ gha 4	*yâ va* 0	*kâ vayâ.bhâ* 12

corresponding to our $8x^3+4x^2+10y^2x = 4x^3+0x^2+12y^2x$, which is excellent for reducing the equation and may also be an adequate means to express a resolving algorithm once such an algorithm is known; but it does not allow, for instance, that *yâ* (the first unknown) to the third power be replaced by *P gha*, where *P* is itself a polynomial. In other words, the notation does not allow embedding, the replacement of a simple mathematical object by a different, complex object – *the* essential feature, if any exists, of the change which affected mathematics so thoroughly after 1600.

As we shall see, schematic notations also developed in European (and Maghreb) algebra, but they were eventually abandoned as a main means of expression. On the other hand, elementary embedding began independently of the use of abbreviations. Rather than stages, we should therefore speak of aspects of the expression of algebraic thought, aspects which only to some extent are sequentially ordered.

AL-KHWĀRIZMĪ'S ALGEBRA

Al-Khwārizmī's algebra was purely rhetorical. It dealt with a quantity called *māl* (literally a "possession" or "amount of money", becoming *census* in Latin), its square root (*jidhr*) and *number* (treated as a number of *dirham*, becoming *dragmas* in Latin).

[3] Here and in what follows, all translations into English are mine if nothing else is indicated.

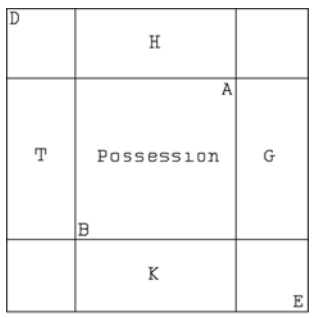
Already in al-Khwārizmī's treatise, however, the *māl* is explained as a number multiplied by itself, and the *jidhr* is identified with the *šay*'/"thing".[4] These terminological complications are traces of a complex prehistory, which does not concern us here, and which can anyhow only be reconstructed hypothetically.

Al-Khwārizmī's algebra proper[5] contains rules for solving reduced first- and second-degree problems ("cases" in what follows), geometric proofs for the correctness of the rules for the mixed cases ("possession and roots are made equal to number" etc.), rules and proofs for the calculation with square roots and binomials, and examples showing how to reduce other problems. The rule for the first mixed case runs as follows:[6]

> But possession and roots that are made equal to a number is as if you say, "A possession and ten roots are made equal to thirty-nine dragmas". The meaning of which is: from which possession, to which is added ten of its roots, is aggregated a total which is thirty-nine? The rule of which is that you halve the roots,[7] which in this question are five. Then multiply them by themselves, and from them 25 are made. To which add thirty-nine, and they will be sixty-four. Whose roots you take, which is eight. Then subtract from it half of the roots, which is five. There thus remain three, which is the root of the possession. And the possession is nine.

In modern symbols: if $y+10\sqrt{y} = 39$ (or $x^2+10x = 39$), then $\sqrt{y} = \sqrt{39+(\frac{10}{2})^2} - \frac{10}{2}$.

Two geometric proofs are given for the correctness of the rule. The first [Hughes 1986: 236*f*] runs as follows:



> A possession and ten roots are made equal to thirty-nine dragmas. Make therefore for it a quadratic surface with unknown sides, which is the possession which we want to know together with its sides. Let the surface be *AB*. But each of its sides is its root. And each of its sides, when multiplied by a number, then the number which is aggregated from that is the number of roots of which each is as the root of this surface. Since it was thus said that there were ten roots with the possession, let us take a fourth of ten, which is two and a half. And let us make for each fourth a surface together with one of the sides of the surface. With the first surface, which is the surface *AB*, there will thus be four equal surfaces, the length of each of which is equal to the root of *AB* and the width two and a half. Which are the surfaces *G*, *H*, *T* and *K*. From the root of a surface with equal and also unknown sides is lacking that which is diminished in the four corners, that is, from each of the corners is lacking the multiplication of two and a half by two and a half. What is needed in numbers for the quadratic surface to be completed is thus four times two and a half multiplied by itself. And from the sum of all this, twenty-five is aggregated. [...].

[4] I shall italicize the word "thing" when it is used as an algebraic unknown; below, when discussing the Italian material, also other powers and "number" when occurring as "power 0".

[5] This leaves out the chapters on the rule of three, on geometry and on inheritance calculation. The twelfth-century Latin translations also left out the latter two.

[6] I translate (as literally as possible) from Gherardo of Cremona's Latin translation [ed. Hughes 1986: 234*f*], arguably a better witness of the original text than the extant Arabic manuscripts – see [Høyrup 1998] and [Rashed 2007: 86–89].

[7] That is, the number of roots – in our terms, their coefficient.

In consequence, the argument continues, the area of the completed square *DE* is 39+25 = 64, and its side 8. Subtracting $2\cdot2\frac{1}{2} = 5 = {}^{10}/_{2}$ we find that the side of *AB* is 8–5 = 3.

Al-Khwārizmī's illustrates the use of the technique (here the rule for "possession and number is made equal to roots") by this example:

> "Divide ten in two parts, and multiply each of them by itself, and aggregate them. And it amounts to fifty-eight". Whose rule is that you multiply ten minus a *thing* by itself,[8] and hundred and a possession minus twenty *things* results. Then multiply a *thing* by itself, and it will be a possession. Then aggregate them, and they will be one hundred, known, and two possessions minus twenty *things*, which are made equal to fifty-eight. Restore then one hundred and two possessions with the *things* that were taken away, and add them to fifty-eight. And you say: "One hundred, and two possessions, are made equal to fifty-eight and twenty *things*". Reduce it therefore to one possession. You therefore say: "Fifty and a possession are made equal to twenty-nine and ten *things*". Oppose hence by those, which means that you throw twenty-nine out from fifty. There thus remains twenty-one and a possession, which is made equal to ten *things*. Hence halve the roots, and five result. [...].

In symbols (replacing the *thing* by *x*): Given is $10 = x+(10–x)$ and $(10–x)^2+x^2 = 58$. Therefore, stepwise, $100+x^2–20x+x^2 = 100+2x^2–20x = 58$; $100+2x^2 = 58+20x$; $50+x^2 = 29+10x$; and finally $21+x^2 = 5x$, the reduced equation for which we have a rule. As far as al-Khwārizmī's technique goes, it thus agrees with what *we* would do; but as we see, the composite expression $(10–x)^2$ has to be expanded before it can be inserted into the equation, there is room for no other way to operate with it.[9]

THE BEGINNING OF ABBACUS ALGEBRA

In 1202, with revision in 1228, Leonardo Fibonacci wrote his *Liber abbaci*, which contains a final section on algebra. As I have argued elsewhere [Høyrup 2005], Fibonacci must have known (and drawn part of his material from) an environment somewhat similar to the Abbacus school as we know it from Italy from the later 13th century onward (see imminently), located probably in the Western Islamic region (the Maghreb and Islamic Spain), Catalonia and Provence. However, his *algebra* is quite different from what we find in Italian Abbacus writings and close to al-Khwārizmī in style (though wider in range, being also influenced by Abū Kāmil).

The earliest traces of the Abbacus school turn up in the sources around 1265. It was primarily frequented by merchant and artisan youth for c. two years (around the age of 11), who were taught the mathematics needed for commercial life: calculation with the Hindu-Arabic numerals; the rule of three; how to deal with the complicated metrological and monetary systems; alloying; partnership; simple and composite

---

8 The previous example – also of type "divided 10" – has already made the position that one part is represented by a *thing*, whence the other must be 10 minus 1 *thing*.

9 Al-Khwārizmī thus would have had great troubles to make his reader follow the calculation $(10–x)^2+(10–x)x = (10–x)\cdot(10–x+x) = (10–x)\cdot10 = 100–10x$, so easy when symbols allow us to treat $10–x$ as a simple entity.

discount; the use of a "single false position"; and area computation. Smaller towns might employ a master, in towns like Florence and Venice private Abbacus schools could flourish. In both situations Abbacus masters had to compete, either for communal positions or for the enrolment of students.

Algebra was not part of the school curriculum, but from the early 14th century it turns up (together with other techniques like the "double false position" that were too difficult for normal students) in a number of abbacus texts. Such matters may have been meant for the education of apprentices working also as assistants, but at least algebra functioned as a token of professional aptitude and therefore also enjoyed high prestige.

The earliest extant abbacus book containing a presentation of algebra is Jacopo da Firenze's *Tractatus algorismi*, written in Montpellier in 1307, in Tuscan Italian in spite of its Latin title.

It is *not* derived from Fibonacci's algebra, nor from the "scholarly" level of Arabic algebra – that of al-Khwārizmī, Abū Kāmil, al-Karaji and Ibn al-Bannā' – but probably from a level integrated with commercial teaching. However, the total absence of Arabicisms shows that the direct source must have been located in a Romance-speaking region – the best guess appears to be a Catalan environment of Abbacus-school type.[10]

Jacopo's algebra is also purely rhetorical, but it differs that of al-Khwārizmī in several ways: whereas the second power is referred to as *censo* (now with all connotations of money forgotten), the first power is never the "root" but invariably the *cosa*/*thing*, and the number term is always spoken of as *numero*, never as an amount of money.[11] Half of the examples (all for the first and second degree) also deal with (varied but invariably sham) commercial problems, which are almost absent from al-Khwārizmī and Fibonacci,[12] and uses the rule of three as a tool in certain algebraic arguments. As if he were conscious of introducing a new field, Jacopo avoids all abbreviations of algebraic core terms (even though non-algebraic words are often abbreviated, as habitual in manuscripts from the epoch).

Al-Khwārizmī only treats problems of the first and second degree. Problems of higher degree turn up in Abū Kāmil and Fibonacci but are not treated systematically. Jacopo instead gives rules for such basic "cases" of the third and fourth degree as are homogeneous or can be reduced to the second degree, forgetting only two biquadratics.[13] Examples accompany rules for the first and second degree only.

---

[10] For this and what follows about the beginning of abbacus algebra, see [Høyrup 2006] or [Høyrup 2007a: 147–182]

[11] Both *roots* and *dragmas* used in this way turn up (together with geometrical proofs) in a few 15th-century abbacus manuscripts of encyclopedic character, whose authors show explicit interest in the founding fathers of the field. But even in their case this pious service is isolated from their own use of algebra.

[12] Actually, they deal with only one type: A given amount of money is divided first among an unknown number (say, x) of persons, and afterwards between $x+N$ persons (N given). The sum of or the difference between the shares in the two cases is also given.

[13] Since al-Karajī, such problems had been solved routinely and systematically in Arabic algebra.

We may look at the first and simplest of the two examples for the first-degree case, "*things* are equal to *number*":[14]

> make two parts of 10 for me, so that when the larger is divided by the smaller, 100 results from it. Do thus, posit that the larger part was a thing. Hence the smaller will be the remainder until 10, which will be 10 less a *thing*. And thus we have made two parts of ten, of which the larger is a *thing*, and the smaller is 10 less a *thing*. Now one shall divide the larger by the smaller, that is, a *thing* by 10 less a *thing*, from which shall result 100. And therefore one shall multiply 100 times 10 less a *thing*. It makes 1000 less 100 *things*, which equal one *thing*. [...].

This draws on the same cognitive resources as al-Khwārizmī's text (without the proofs).

THE IMMEDIATE SUCCESSORS

During the following decades, algebra turns up in a number of abbacus books, sometimes in more or less general expositions, sometimes as isolated problems. The most interesting early exposition is in Paolo Gherardi's abbacus treatise (Montpellier, 1328).[15] Most striking here is the appearance of irreducible third-degree cases, solved by means of false rules – glaringly false indeed for anybody understanding the matter.[16] These false rules survived for more than 200 years (they are still in Bento Fernandes' *Tratado da arte de arismetica* from 1555 [Silva 2006]). They probably served to outdo colleagues in the competition for positions and students; their survival is strong evidence that few abbacus teachers understood much of algebra. Whether Gherardi understood is doubtful; indirect evidence shows that he did not invent the wrong rules.

Less conspicuous but also of importance is the earliest use of a diagram for a formal calculation (missing in the actual manuscript, which is a copy, but described unambiguously in the text)

$$\begin{array}{lcl} 100 & \diagdown\!\!\!\!\diagup & 1\ \text{cosa} \\ 100 & & 1\ \text{cosa piu 5.} \end{array}$$

It turns up in a pure-number-version of the problem described in note 12, which we may translate $\frac{100}{x}+\frac{100}{x+5}=20$. It implies an understanding of the operations (cross-multiplication etc.) needed to add the formal fractions[17] $\frac{100}{1\,cosa}$ and $\frac{100}{1\,cosa\ piu\ 5}$.

In a *Trattato dell'alcibra amuchabile* from c. 1365, we find such formal fractions written out repeatedly - for instance, in the same problem, $\frac{100}{by\ a\ thing}\ \frac{100}{by\ a\ thing\ and\ plus\ 5}$ [ed. Simi 1994: 42], explained to be performed "in the mode of a fraction" and explained

[14] [Høyrup 2007a: 304*f*].

[15] The complete text is in [Arrighi 1987], the algebra chapter with English translation in [Van Egmond 1978].

[16] For instance, the case "*cubes* equal to *things* and *number*", solved according to the rule for "*censi* equal to *things* and *number*". For the mathematically thoughtful this should imply that the *cube* is equal to the *censo*, and by division (another rule given by Gherardi) that the *thing* equals 1. Direct easy check was barred by the appearance of radicals in the solution.

[17] These are "formal" in the sense that the form of the fraction is taken not to express an actual broken number but the ratio between algebraic expressions.

in analogy with $\frac{24}{4}+\frac{24}{6}$. We are thus presented with a rudimentary example of *symbolic operation (including embedding) without abbreviation*.[18]

The expression “by a *thing* and plus 5” (*per una cosa e più 5*) is mirrored elsewhere (p. 50) in a similar fraction, where the denominator is “by two *things* and less 6” (*per due cose e meno 6*). They show that the author operated with a notion of additive and subtractive numbers, and that a subtraction is understood as the addition of a subtractive number. We should not identify the subtractive numbers with negative numbers, since they cannot occur as results; but the idea was close at hand (and soon grasped).

We also finds schemes for the multiplications of binomials (consisting of number and irrational root), for instance (p. 18) for (5+√20)·(5–√20):

5 and plus ℜ of 20

times

5 and less ℜ of 20

Sometimes, crossing lines showing the cross-multiplication replace the word “times” (*via*) – or both occur. The same lines are used in earlier abbacus manuscripts when the multiplication of mixed numbers is shown.

The *Trattato dell'alcibra amuchabile* copies Jacopo's algebra verbatim, but also has most of Gherardi's false solutions in a version which appears to predate Gherardi; all of this material must thus go back to before 1330 and hence precede Giovanni di Davizzo's algebra (from 1339, and known only from a fragment included in a manuscript from 1424) and the *Aliabraa argibra*, written by one Dardi of Pisa in 1344.

Though independent of Jacopo, Giovanni gives almost the same rules (and one false rule, almost fully illegible in the manuscript but not one of Gherardi's). However, he also gives correct examples for calculation with square roots and binomials consisting of rational numbers and roots – mostly roots of square numbers, but treated as if the roots were irrational, and not taking advantage of the possibility which this choice offers for checking (edition and translation of the relevant part in [Høyrup 2007c: 479–481]). Even more striking, he teaches the multiplication of powers (which allows us to see how these are labelled) and the division of lower by higher powers.

The powers are composed multiplicatively – the *censo of cube* is the fifth power, the *cube of cube* the sixth, etc. This is wholly traditional, both Diophantos and al-Karajī do the same. In Greek and Arabic, no linguistic problem inheres in this, but the Italian (and corresponding Latin) genitive construction soon became a challenge by suggesting embedding instead of multiplication: the *cube* of 2 is 8, and the *cube* of 8 is 512 – but the *cube of cube* of 2 is 64!

---

[18] The idea was borrowed from Maghreb mathematics – [Djebbar 2005: 93] shows in facsimile an equation in a manuscript containing a fraction $\frac{48}{\frac{1}{2}\ t}$.

Giovanni did not see the problem, and made worse in his division of lower by higher powers; here the power $-n$ is replaced by the nth root (by *number* if $n = 1$), and even roots are composed multiplicatively ("dividing *number* by *cube of cube* gives *cube root of cube root*", etc.). Giovanni is likely to have invented this system himself – there are no traces of it except in writings which repeat it wholesale; such wholesale repetitions, on the other hand, circulate until Bento Fernandes. We may conclude, firstly, that the ambition to extend the reach of algebra (whether intellectual or career ambition we may leave aside) was not restricted to the production of false solutions; and secondly, that few Abbacus masters had the least need for higher roots, and that most of them therefore did not need to discover the problem. We may also observe that Giovanni's extension, a dead-end as it is, was guided by an intuitive idea that mathematics (but unfortunately the mathematics he already knew about) must be coherent.[19]

Dardi's treatise is at a different level; anybody with some mathematical training who reads it will feel that here a genuine mathematician is speaking. What first strikes one is that he solves 194 cases correctly[20] – a number he reaches by involving radicals (square and cube roots of numbers as well as algebraic powers). He also gives rules for solving four "irregular" cases of the third and fourth degree, rules which only hold under particular circumstances (as he points out), but which may still serve (namely in a competition, we may add). The rules had been guessed (apparently not by Dardi) through a change of unknown in homogeneous problems;[21] deriving them requires a good understanding of the algebra of polynomials (see [Høyrup 2007b: 6*f*]); even Dardi's own elimination of radicals requires good insight.

However, Dardi's work is interesting not only as evidence of level. He uses abbreviations consistently not only for *radice* ("root", which I shall render $\Re$) but also for the *thing* and the *censo* – *c* and *ç*, respectively. At the same time, he uses the fraction model in a way which bars the development of formal calculations – seeing $/_4$ in $^3/_4$ as a *name* ("fourths") and not as an operation, he generalizes and writes, e.g., 10 *things* as $\frac{10}{c}$. In spite of his having schemes similar to those of the *Trattato di alcibra amuchabile*, Dardi's style is thus a good example of *syncopation not pointing toward symbolic calculation*.

Seen under a different angle, his treatise agrees more thoroughly than most abbacus algebras with the idea that mathematics should be built on arguments. He gives geometric proofs, ultimately based on those of al-Khwārizmī but as different from these in details as if he had seen them once and then reconstructed them from memory; he certainly did not copy directly. He uses the rule of three to show how to divide by a binomial (3+√4 – Dardi also uses rational roots "as if they were surds", and is indeed the one who uses this phrase; even he takes no advantage of the choice). Finally, he

[19] In the mid-15th-century encyclopedias mentioned in note 11 we find a better system, drawing on formal fractions; they speak of "fraction denominated by *censo*" etc.

[20] Or almost so, cf. [Van Egmond 1983: 417]. One solution asks for a fifth and one for a seventh root. Having no adequate terminology Dardi replaces them by "cube root" and "root of root", respectively, although he understands the embedding of root taking perfectly in other places.

[21] For instance, regarding a capital which grows in three years from 100 £ to 150 £, taking as *thing* not the value of the capital after one year but the rate of interest.

gives an intuitive proof of the sign rule "less times less makes plus", based on the calculation $(10–2)\cdot(10–2) = 10\cdot10–2\cdot10–2\cdot10+(–2)\cdot(–2)$,[22] arguing that it should be 64, which it is indeed if $(–2)\cdot(–2)$ is 4 but not if it is (–4) or 0.

In the end we may take note that Dardi does not like the ambiguous expression "the *censi*" when he wants to refer to their coefficient; instead he speaks of "the quantity of *censi*". This step toward terminological precision is likely to be his own invention; it had no perceptible impact.

ALTERNATIVES TO THE FALSE RULES

I have neither time (in my presentation) nor space (in writing) to discuss more treatises in detail. Instead I shall arrange the discussion according to select themes, beginning with the false rules. A manuscript from the outgoing 14th century [ed. Franci & Pancanti 1988: 98] speaks of the existence of particular roots beyond the square and cube roots, and explains one called "cube root with addition". The "cube root of 44 with addition of 5" is told to be 4, because $4^3 = 44+5\cdot4$ – in general, the "cube root of n with addition α" is t if $t^3 = n+\alpha t$. This is one of the equations provided with a false solution by Gherardi; the inventor of this roo+t thus knew that Gherardi's solution was false, and wanted to do better. The author of the present manuscript is not impressed; he observes that this root mostly does not exist (as an integer). He points out, however, that the cases $t^3+\beta t^2 = m$, $t^3 = \beta t^2+m$ and $\beta t^2 = t^3+m$ can be reduced to the form $t^3 = n+\alpha t$ and thus be solved by means of the same root – showing also that solutions may exist even if n is "a debt", i.e., a negative number. The way he expresses the coefficients of the transformed equations shows that he went through exactly the same change of variable as we would.

The manuscript does not identify the other particular roots, but one of them is probably the "pronic root" which we encounter in a number of sources. If $t^4+t = N$, then some sources (e.g. Pacioli [1494: I, 115^v] identify t^2 as the pronic root, others [e.g. Pierpaolo Muscharello [ed. Chiarini et al 1972: 163]) state it to be t. Benedetto da Firenze [ed. Pieraccini 1983: 26] mentions it in 1463 in connection with the equation $x^2+\sqrt{x} = 18$ but does not make it clear whether x or $t = \sqrt{x}$ should be the pronic root. What he does make clear is that even this root served to "solve" irreducible equations.

Pacioli [1494: I, 150^r] states that so far only equations where the three powers involved are "equidistant" had been correctly solved. He may have known about the solution of other equations by means of these particular roots (he admits that certain other equations can be solved *a tastoni*, "by feeling one's way"), but if so he did not see them as genuine solutions. With hindsight we would say that he was right – but with the proviso that the *transformations* that go together with the "cube root with addition" were exactly those which permitted Cardano to solve cubic equations *in general* after having solved cases with no second-degree term.

[22] "–2" is still to be understood as a subtractive, not a negative number. When repeating the same proof, Luca Pacioli [1494: I, 113^r] instead thinks of genuine negative numbers. He finds them "absurd" *but necessary* – the quest for coherence had enforced expansion of the number concept.

NAMES FOR POWERS AND ROOTS

The contradiction multiplication/embedding in the construction of names for powers and roots was eventually productive, but at first a cause for confusion.

Changes in the terminology for *roots* set in first, perhaps because the problem was most obvious here. Dardi, as mentioned, understood the embedding of roots but stumbled on the ensuing lack of terminology for the fifth and seventh root. Our earliest evidence for the term *radice relata*, "related root" for the fifth root is Antonio de' Mazzinghi, a mathematically brilliant abbacus teacher who probably died rather young in 1385.[23] Later, this became "first related root", the "second related root" being $^7\sqrt{}$, the "third related root" being $^{11}\sqrt{}$, etc.[24] Other roots were then named by embedding. As we see, the system is consistent, but quite unhandy.

The earliest evidence for (ambivalent) naming of *powers* by embedding is in the manuscript which speaks of particular roots, and which starts by presenting the powers until the sixth, including products which remain within this limit [ed. Franci & Pancanti 1988: 3–5]. The author is aware that the powers are in continuous proportion and uses this in his arguments, but apart from that the explanation is confusing (but not necessarily confused) – perhaps because the author is moving on unfamiliar ground. The *thing* multiplied by it self is said to be

> a root which is called a *censo*, so that it is the same to say a *censo* as to say a quantity which has a root, born from a number multiplied by itself, so as it would be to say that if the *thing* produces 4 in number, the *censo* should produce the square of the *thing*, that is, what 4 multiplied by itself makes, that is that the value of the *censo* will be 16, so that, seen that 4 is the root of 16, it therefore comes that the *thing* is said to be the root of the *censo*, so that it is as much to say *censo* as root of number.

The mixing-up of *having* and *being* a root goes through the whole discussion, but the consistently correct numerical examples suggest that the confusion is merely or principally in the words, not in the underlying thinking.[25] The product of a *thing* and a *censo* is called a *cube* (and "a cubic root of a given number"), the product of a *thing* and a *cube* is a "*censo of censo*, which is to say the root of the root of a given quantity"; the explanation of the numerical example suggests that the name is understood through embedding. *Thing* times *censo of censo* is said to be

> *cube of censi*, which is as saying a root born from a square quantity multiplied by a cube quantity [...]; and some call this root related root. So that it would be the same to say *cube of censo* as related root of a given quantity.

---

[23] [Ulivi 1996: 109–115]. We know his writings through extracts in the encyclopedic works referred to in note 12.

[24] This terminology, though used for a particular purpose, is in [Pacioli 1494]; cf. presently.

[25] The underlying idea *may* be that since a *thing* is also called a *root*, the higher powers must also be "roots" of some kind. If this explanation is correct, we may understand "cubic root of a given number" (etc.) as "cubic «root» *on* a given number".
Jean Peletier [1554: 5], somehow knowing the usage, explains it by speaking of the powers as "*nombres radicaux*, that is, which have in themselves some root to extract".

Here, the thinking is obviously multiplicative. The next step, however, goes by embedding: a *thing* times a *cube of censo* is

> a *censo of cube*, which means as much as saying, taken the root of a quantity, and of this quantity taken its cube root, so that if the *thing* is 3, the *censo* will be 9, the *cube* will be 27, the *censo of censo* will be 81, the *cube of censo* will be 243, the *censo of cube* will be 729, because taken the root of 729 will make 27, whose cube root is 3 and equal to the value of the *thing*.

Then products of powers are discussed – and unfortunately it is said in the end that *cube* times *cube* is "*cube of cube*, that is cube root of cube root".

What looks like a further development of this system is described twice by Pacioli [1494: I, 67^{v}, 143^{r-v}]. In the interest of completeness (i.e., describing a system he has not invented and does not use) he gives in parallel the habitual sequence of names (now based on embedding) and the "root names", which are now completely arithmetized. The former are

> *number – thing – censo – cube – censo of censo – first related – censo of cube and also cube of censo – second related – censo of censo of censo – cube of cube – ...*,

ending with the 29th power, the *ninth related*. The corresponding root names are *1st root*, *2nd root*, ... *30th root*. Since *thing*$^{n-1}$ is the *n*th *root*, this arithmetization, while adequate for seeing which terms (in Pacioli's expression) are "equidistant" and thus for reducing equations, e.g., of the type $x^{2+p}+\alpha x^{1+p} = \beta x^{1+p}$, they are less useful for seeing for instance that the type $x^{2p}+\alpha x^{p} = \beta$ is of the second degree in x^{p}.[26]

A more adequate arithmetization came from the abbreviated writing of equations, mostly occurring in the margins of manuscripts – for instance Vatican, Vat. lat. 3129, written by Pacioli in 1478. Here, abbreviations for powers (*co* for *cosa*, ☐ alternating with *cen* for *censo*) are written above or as superscript following the coefficient.[27] This graphic distinction allowed first Chuquet (in 1484 [ed. Marre 1880: 632 and *passim*]) and later Bombelli [1572] to replace the abbreviation by the number of the power[28] – Bombelli with an arc below[29] to further emphasize the graphic distinction.[30] Both use the numbers we regard as exponents.[31]

---

[26] When needing on Fol. 182^{r} the sequence of genuine roots in problems about composite interest (and not reporting what he had found in circulation), Pacioli still uses the multiplicative system for everything except $^{5}\surd$ – in order, "ℜ" ($\surd$), "cube ℜ" ($^{3}\surd$), "ℜℜ" ($^{4}\surd$), "related ℜ" ($^{5}\surd$), "cube ℜ of cube ℜ" ($^{6}\surd$), "ℜℜ of cube ℜ" ($^{7}\surd$), "ℜ of cube ℜ of cube ℜ" ($^{8}\surd$), etc. One wonders how deep his understanding was.

[27] Even this vertical organization goes back to Maghreb algebra – see, e.g., [Cajori 1928: I, 93*f*] and [Djebbar 2005: 92].

[28] Chuquet's sense of system also lets him designate $^{n}\surd$ as ℜ.^{n} (even when $n = 2$).

[29] In the manuscript, the exponent is above the coefficient and the arc separates the two – facsimile in [Bombelli 1966: xxxiii].

[30] Tartaglia [1560: 2^{r}] has a table similar to that of Pacioli but with numbering of the *dignitates*/"powers" coinciding with our exponents. He uses the same traditional names (composed with embedding) as Pacioli. However, he is preceded by Stifel [1544: 235^{r}–237^{r} closely followed by Peletier [1554: 8–11], who speak of the numbers as *exponentes*/*exposans*.

[31] Eventually, when combined with Viète's use of letters, this led to the modern notation of variable with exponents.

SCHEMES

The use of abbreviations for frequently recurrent words or endings was common in manuscripts from the period. The abbreviation of *cosa*, *censo*, *radice* etc. thus adopted a tool which already existed.[32] So do the schemes for multiplication of binomials which we encountered in the *Trattato dell'alcibra amuchabile* and in Dardi (which borrow the cross indicating how to multiply mixed numbers) as well as the formal fractions with algebraic expressions in the denominator.

A final development exemplifying the principle of algebraic wine in non-algebraic bottles is the emergence of algebraic calculation within schemes. The manuscript Vatican, Ottobon. lat. 3307, Fol. 331^{r} (c. 1465) contains a problem $\frac{100}{1\rho}+\frac{100}{1\rho+7}=40$ (the formal fractions, without + and =, are already in the text; ρ is used for *thing*). The solution makes use of the transformation $\frac{100\rho+100\cdot(\rho+7)}{(1\rho)\cdot(1\rho+7)} = \frac{100\rho+(100\rho+700)}{1\sigma+7\rho} = 40$, whence $200\rho+700 = 40\sigma+280\rho$ (σ is used for *censo*). In the margin, the solution is summarized as follows:

$$
\begin{array}{llll}
100\rho & & & \\
\underline{100\rho} & \underline{700} & & \\
\underline{200\rho} & \underline{700} & & \\
\;\;1\sigma & \underline{7\rho\;\;\;\;\;} & 40 & \\
 & & & \\
200\rho & 700 & \text{———} & 40\sigma\ \langle 280\rho\rangle
\end{array}
$$

("280ρ" has been forgotten but stands in the text). This emulates the way non-algebraic items can be added, combined with the fraction notation. The stroke -, seemingly an equation sign, is also used more broadly for confrontations – thus confronting (fol. 338^{r}) the contributions of two business partners. Since Regiomontanus [ed. Curtze 1902: 278] uses exactly the same scheme, it is likely to represent a common procedure.

In the late 14th-century manuscript introducing the cube root with extension we find not only the abbreviations $\Re$ (*radice*), p (*più*/"plus"), m (*meno*/"less"), ρ (*cosa*) and *c* (*censo*) and Dardi's diagram for the multiplication of binomials but also [ed. Franci & Pancanti 1988: 11] a scheme for multiplying longer polynomials which follows the principles of number multiplication *a chaselle* with vertical columns. Similar schemes are not only found in quite a few later abbacus algebras; they also came to play an important role in Stifel's *Arithmetica integra* [1544: Fol. 123–125 und *passim*], in Scheubel's *Algebrae compendiosa facilísque descriptio* [1551: 3^{v}*ff*], in Peletier's *L'algebre* [1554: 15–22] and in Ramus's *Algebra* [1560: A iii^{r}].

[32] As non-algebraic abbreviations, those for *cosa* and *censo* were rarely used systematically (Dardi being an exception). Only *radice* was used almost consistently in certain manuscripts.
The use of abbreviations may have received inspiration from Maghreb algebra. Here, however, single-letter abbreviations were employed, inside a fully consistent notation. *If* inspired, the Italian writers understood the Maghreb abbreviations within the framework of their own habits.

THE EFFECT OF A CHANGE OF AIR

Fraud and experiments with not immediately reducible higher-degree equations, not too consistent use of abbreviations, and schemes for the calculation with polynomials – this is more or less as far as Italian abbacus algebra went before 1500. Only on the first, somewhat dubious account did it go beyond developments that had already taken place in the Maghreb well before 1250 – developments which some abbacus authors had probably known about directly or indirectly, but which the abbacus environment had to digest before it could make them their own

The experiments with higher-degree equations led to a general breakthrough due to del Ferro, Tartaglia, Cardano and Ferrari in the years 1515–1545. The use of abbreviations and schemes also took root for good in the sixteenth century – beginning however in Germany already in the 15th century,[33] well before it happened in the Italian environment, and also soon to be seen in French writings. In consequence, writings on algebra already look very different from 15th-century Italian predecessors well before Viète.

We may ask why. Book printing *per se* is hardly the explanation – in the manuscript version of Bombelli's *L'algebra*[34], the symbolism for powers and parentheses is different from what we find in the printed edition from [1572], and actually more transparent. Neither is the mere migration to new territories likely to explain much, since the new trends can also be seen in Italy. We may notice, however, that the innovations go together with integration of the abbacus environment with environments more oriented toward university learning – del Ferro was a University professor, Cardano a most learned physician, German algebra was expressed in Latin already in the 15th century.[35] Already Chuquet, in many respects (an unsuccessful) precursor of 16th-century developments, was actually a university scholar, having completed the degrees of the arts as well as medicine in Paris. The Italian abbacus 14th and 15th-century abbacus environment, though governed by norms of precision and coherence at the levels where every abbacus master and any good student could understand what went on [Høyrup 2007b], lacked a social mechanism which could impose intellectual progress on everyone once it had been made (*vide* the survival of the fraudulent rules and Giovanni's nonsensical divisions for more than 200 years). Such mechanisms were not perfect in the 16th century (nor today), but much stronger than in the free-market teaching in Italy in the 14th and 15th centuries. Once Stifel had published his *Arithmetica integra* in [1544], it was obvious to both Peletier (who cites him) and Ramus (who pretends never to have heard about him in [1560] as well as [1569]) to draw on the inspiration he offered. Here, of course, printing was important: it was much easier to have access to the good model; who like Fernandes [Silva 2006] took his inspiration from the manuscripts he could get hold of depended on good or bad luck.

[33] The earliest evidence for fully systematic use of standard abbreviations may be the appendix to Robert of Chester's translation of al-Khwārizmī's *Algebra* [ed. Hughes 1989: 67]. Schemes come later, for instance in Christoff Rudolff's *Coss* [1525].

[34] A facsimile of a representative page is in [Bombelli 1966: xxxiii].

[35] See [Folkert 2006: XII]. In [Høyrup, forthcoming] I argue that the role Folkerts ascribes to Regiomontanus is overstated – Regiomontanus turns out to be very close to Italian models and no more systematic than these.

The 16th-century maturation and stabilization of formal fractions, names for powers and roots organized with embedding (or arithmetized), abbreviations for operations used every time and not just now and then, and schemes – all developments starting in the 14th and 15th century on the basis of existing non-algebraic writing – made possible that freer development of the algebraic language which set in with Viète and Descartes, and in the end reduced the schemes – for a while the most advanced expression of the autonomy of algebra from spoken language – to algorithmic aids or eliminated them altogether from the algebra textbooks.

References

Arrighi, G. (ed.) (1987). Paolo Gherardi, *Opera Mathematica: Libro di ragioni – Liber habaci*. Lucca: Pacini-Fazzi.

Bombelli, R. (1572). *L'Algebra*. Bologna: Giovanni Rossi.

Bombelli, R. (1966). *L'Algebra*. 1. ed. Integrale. Milano: Feltrinelli.

Cajori, F. (1928). *A History of Mathematical Notations*. 2 vols. La Salle, Illinois: Open Court.

Chiarini, Giorgio, et al (Eds.) (1972). [Pierpaolo Muscharello], *Algorismus*. Verona: Banca Commerciale Italiana.

Cuomo, S. (2001). *Ancient Mathematics*. London & New York: Routledge.

Curtze, M. (Ed.) (1902). *Urkunden zur Geschichte der Mathematik im Mittelalter und der Renaissance.* 2 vols. Leipzig: Teubner.

Datta, B. & Singh, A. N. (1962). *History of Hindu Mathematics. A Source Book.* 2 vols. Bombay: Asia Publishing House.

Djebbar, A. (2005). *L'Algèbre Arabe: Genèse d'un Art*. Paris: Vuibert.

Folkerts, M. (2006). *The Development of Mathematics in Medieval Europe: The Arabs, Euclid, Regiomontanus*. Aldershot: Ashgate.

Franci, R. & Pancanti, M. (Eds.) (1988). Anonimo (sec. XIV), *Il Trattato d'Algibra*. Siena: Servizio Editoriale dell'Università di Siena.

Høyrup, J. (1998). "`Oxford' and `Gherardo da Cremona': On the Relation between two versions of al-Khwārizmī's algebra". In *Actes du 3me Colloque Maghrébin sur l'Histoire des Mathématiques Arabes* (vol. II. pp. 159–178). Alger: Association Algérienne d'Histoire des Mathématiques.

Høyrup, J. (2005). "Leonardo Fibonacci and *Abbaco* culture: A Proposal to invert the roles". *Revue d'Histoire des Mathématiques*, *11*, 23–56.

Høyrup, J. (2006). "Jacopo da firenze and the beginning of Italian Vernacular algebra". *Historia Mathematica*, *33*, 4–42.

Høyrup, J. (2007a). *Jacopo da Firenze's Tractatus Algorismi and Early Italian Abbacus Culture*. (Science Networks. Historical Studies, 34). Basel etc.: Birkhäuser.

Høyrup, J. (2007b). "What did the abbacus teachers really do when they (sometimes) ended up doing mathematics?" *Filosofi og Videnskabsteori på Roskilde Universitetscenter.* 3. Række: *Preprints og Reprints* 2007 Nr. 4.

Høyrup, J. (2007c). “Generosity: No doubt, but at times excessive and delusive”. *Journal of Indian Philosophy*, *35*, 469–485.

Høyrup, J.forthcoming. [Review af Menso Folkerts 2006]. To appear in *Aestimatio*.

Hughes, B. B. (Ed.) (1986). “Gerard of cremona's translation of al-Khwārizmī's *Al-Jabr*”. *Mediaeval Studies*, *48*, 211–263.

Hughes, B. B. (Ed.) (1989). Robert of Chester’s latin translation of al-Khwārizmī's *Al-jabr*. Wiesbaden: Franz Steiner, 1989.

Marre, A. (Ed.) (1880). “Le triparty en la science des nombres par maistre Nicolas Chuquet parisien”. *Bullettino di Bibliografia e di Storia delle Scienze Matematiche e Fisiche*, *13*, 593-659, 693-814.

Nesselmann, G. H. F. (1842). *Versuch Einer Kritischen Geschichte der Algebra*. I. *Die Algebra der Griechen*. Berlin: G. Reimer.

Pacioli, L. (1494). *Summa de Arithmetica Geometria Proportioni et Proportionalita.* Venezia: Paganino de Paganini.

Peletier, J. (1554). *L'algebre*. Lyon: Ian de Tournes.

Pieraccini, L. (Ed.) (1983). M^{o} Biagio, *Chasi Exenplari all Regola dell'Algibra Nella Trascelta a Cura di M^{o} Benedetto*. Siena: Servizio Editoriale dell'Università di Siena.

[Ramus, P.] (1560). *Algebra*. Paris: Andreas Wechelum.

Ramus, P. (1569). *Scholarum Mathematicarum Libri Unus et Triginta*. Basel: Eusebius Episcopius.

Rashed, R. (Ed., Trans.) (2007). Al-Khwārizmī, *Le Commencement de l'Algèbre*. Paris: Blanchard.

Rudolff, Ch. (1525). *Behend und Hübsch Rechnung Durch die Kunstreichen Regeln Algebra, so Gemeincklich die Coss Genennt Werden*. Straßburg.

Scheubel, J. (1551). *Algebrae Compendiosa Facilísque Descriptio, qua Depromuntur Magna Arithmetices Miracula*. Paris: Gulielmum Cavellat.

Silva, M. do C. (2006). “The algebraic contents of Bento Fernandes' tratado da arte de arismetica (1555)”. *Preprint*, Centro de Matematica da Universidade do Porto, April 27, 2006. Forthcoming, *Historia Mathematica*.

Simi, A. (Ed.) (1994). Anonimo (sec. XIV), *Trattato dell'Alcibra Amuchabile*. Siena: Servizio Editoriale dell' Università di Siena.

Stifel, M. (1544). *Arithmetica Integra*. Nürnberg: Petreius.

Tartaglia, N. (1556). *La Seconda Parte del General Trattato di Numeri, et Misure.* Venezia: Curtio Troiano.

Ulivi, E. (1996). “Per una biografia di Antonio Mazzinghi, maestro d'abaco del XIV secolo”. *Bollettino di Storia Delle Scienze Matematiche*, *16*, 101-150.

Van Egmond, W. (1978). “The earliest vernacular treatment of algebra: The libro di ragioni of Paolo Gerardi (1328)”. *Physis*, *20*, 155-189.

Van Egmond, W. (1983). “The algebra of master Dardi of Pisa”. *Historia Mathematica*, *10*, 399-421.